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On the freedom of choice of the action-angle variables for Hamiltonian systems

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Abstract. The transformations of the action-angle variables allowed by the definition are described and the arbitrariness in the dependence of the Hamiltonian on the action-angle variables is explained. For Hamiltonian systems with the SU(n) algebra of integrals of motion an inverse relationship of 'actions' to generators of the symmetry group is discussed.

1. Introduction

The idea of introducing the action-angle variables in the study of Hamiltonian systems comes from astronomy. The definition of these variables given in the 19th century by a French mathematician Delaunay, however, has been limited to separable systems. A generalisation, independent of the notion of separability, was formulated by Arnol'd (1978). His approach allows us to define the action-angle variables for every completely integrable Hamiltonian system with n degrees of freedom (i.e. with n independent integrals of motion in involution), provided that the invariant manifolds determined by the integrals are connected and bounded. These manifolds then are diffeomorphic to n-dimensional tori T^n, and the action variables are defined geometrically as the integrals of the differential one-form $\Sigma\, p_i\, dq_i$ over the fundamental cycles Γ_i on the tori: $I_k = \oint_{\Gamma_k} \Sigma\, p_i\, dq_i$. The value of I_k is independent of the choice of the cycle homotopic to Γ_k.

Originally the action-angle variables were applied mainly to perturbation calculus in astronomy. Later they became a useful tool for quantisation of classical systems. The recent discovery of a whole class of completely integrable systems of n particles interacting on the line has renewed interest in the action-angle variables. However, there exist some ambiguities of the choice of the action-angle variables within Arnol'd's definition.

The aim of this paper is to describe the transformations of the action-angle variables allowed by the definition (cf Stehle and Han 1967). For Hamiltonian systems possessing the SU(n) algebra of integrals, an inverse relationship of 'actions' to generators of the symmetry group is discussed, providing the action variables for non-separable systems.

For simplicity, our considerations are limited to two-dimensional systems only, and the generalisation for arbitrary n is briefly discussed.

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2. Preliminaries

We shall consider a classical Hamiltonian system with two degrees of freedom admitting the action-angle variables (I_1, I_2, w_1, w_2). Let $F(I_1, I_2) = (\partial H/\partial I_1)/(\partial H/\partial I_2)$ and let D be an open set of R^2 such that F is defined for all $(I_1, I_2) \in D$ and

(i) $F(I_1, I_2) = \text{constant}$ for all $(I_1, I_2) \in D$ or

(ii) there exists no open set $D' \subset D$ $(D' \neq \varnothing)$ such that $F(I_1, I_2) = \text{constant}$ for all $(I_1, I_2) \in D'$.

In case (i) we shall call the system degenerate (non-degenerate) on D if $F(I_1, I_2)$ is a rational (irrational) number. Continuity arguments ensure that in case (ii) the transformations of the action-angle variables have the same form as for non-degenerate systems.

3. Non-degenerate systems

A classical Hamiltonian system with two degrees of freedom admitting action-angle variables (I_1, I_2, w_1, w_2), where $(I_1, I_2) \in D$, is said to be non-degenerate if the ratio ν_1/ν_2 of the frequencies $\nu_i = \partial H/\partial I_i$ is an irrational number for all $(I_1, I_2) \in D$. In this case each orbit of the system is dense in its torus. This fact is connected with the non-existence of the third independent and single-valued integral of motion. If (I_1, I_2, w_1, w_2) are the action-angle variables, then the uniqueness of the tori restricts the permissible transformations of the action-angle variables to

$$\begin{aligned} J_k &= J_k(I_1, I_2), \\ u_k &= a_{k1}(I_1, I_2)w_1 + a_{k2}(I_1, I_2)w_2 + f_k(I_1, I_2), \end{aligned} \qquad k = 1, 2, \tag{1}$$

where the mapping $(I_1, I_2) \to (J_1, J_2)$ is a diffeomorphism and a_{kl}, f_k are arbitrary C^2 functions such that the matrix $[a_{kl}]$ is non-singular for all $(I_1, I_2) \in D$. We shall limit ourselves to the transformations for which $f_k(I_1, I_2) = 0$. This is justified because the functions f_k do not influence geometrical aspects of the transformation.

The angle variables are infinitely-many-valued functions of the physical variables (for example, of the separation variables). Two sets of coordinates, which differ by a multiplicity of 2π, describe the same point in the phase space. To satisfy the same condition in the new angle variables a_{kl} have to be integers. An analogous argument for the inverse transformation yields $w_k = \hat{a}_{k1}u_1 + \hat{a}_{k2}u_2$ where $\hat{a}_{kl}$ are also integers. The matrix $[\hat{a}]$ is the inverse of $[a]$.

The determinant of the integer matrix is integral, and therefore from $\det([a]^{-1}) = (\det[a])^{-1}$ for integer matrices $[a]$ and $[a]^{-1}$ we obtain $\det[a] = \pm 1$. Conversely, the fact that $[a]$ is the integer matrix and $\det[a] = \pm 1$ implies that $[a]^{-1}$ is an integer matrix too.

From these considerations we can conclude that if $u_k = a_{k1}w_1 + a_{k2}w_2$, then u_1, u_2 are angle variables on the torus, if and only if $[a]$ is an integer matrix and $\det[a] = \pm 1$.

The formula $\oint_{\Gamma_i} \mathrm{d}w_j = 2\pi\delta_{ij}$ defines the cycles Γ_i on the torus corresponding to the angle variables w_i.

Let Γ_i' denote a cycle corresponding to the new angle variable u_i. Four relations (defining the cycles) $\oint_{\Gamma_i'} \mathrm{d}u_j = 2\pi\delta_{ij}$ yield

$$\Gamma_1' = \det[a](a_{22}\Gamma_1 - a_{21}\Gamma_2), \qquad \Gamma_2' = \det[a](a_{11}\Gamma_2 - a_{12}\Gamma_1).$$

Now it is easy to calculate new action variables. By definition,

$$J_1 = \frac{1}{2\pi} \oint_{\Gamma_1'} (I_1 \, dw_1 + I_2 \, dw_2) = \det[a](a_{22}I_1 - a_{21}I_2),$$

$$J_2 = \frac{1}{2\pi} \oint_{\Gamma_2'} (I_1 \, dw_1 + I_2 \, dw_2) = \det[a](a_{11}I_2 - a_{12}I_1).$$

Writing $J_k = a'_{k1}I_1 + a'_{k2}I_2$ we obtain $[a'] = ([a]^{-1})^T$ where $[\cdot]^T$ denotes the transposed matrix. The canonical Poisson-bracket relations $\{u_i, J_j\} = \delta_{ij}$ are automatically satisfied.

Let us now return to formulae (1). Clearly, it is always possible to define new variables as

$$J_k = I_k + C_k, \qquad u_k = w_k + f_k(I_1, I_2).$$

From the Poisson-bracket relation $\{u_i, u_j\} = 0$ we obtain $\partial f_2/\partial I_1 = \partial f_1/\partial I_2$, and ultimately we obtain

$$J_k = I_k + C_k, \qquad u_k = w_k + \partial f/\partial I_k, \tag{*}$$

where f is an arbitrary C^3 function of I_1, I_2.

The shifts (*), together with the linear transformations discussed above, describe the whole freedom of choice of the action-angle variables on fixed tori.

The simple generalisation of the above considerations for non-degenerate Hamiltonian systems with more than two degrees of freedom is obvious. The transformations to new action-angle variables are of the same form as in the case $n = 2$, with integer matrices with integral inverses.

4. Degenerate systems

A classical Hamiltonian system with two degrees of freedom, admitting the action-angle variables (I_1, I_2, w_1, w_2), is said to be degenerate on D if $l_1 \, \partial H/\partial I_1 = l_2 \, \partial H/\partial I_2$ for all $(I_1, I_2) \in D$ with l_1, l_2 integers. This formula means that the ratio of frequencies ν_1/ν_2 is rational. One can therefore construct the third independent integral of motion $I_3 = \sin(l_1 w_1 - l_2 w_2)$, which will be a single-valued, bounded function on the phase space. The existence of the third independent and single-valued integral is, in fact, equivalent to the condition of degeneracy.

In the case of degeneracy the orbits of the system are closed on the tori. Therefore we have a freedom of extension of the one-dimensional tori to two-dimensional ones.

Equivalently, we have a freedom of choice of two independent integrals of motion in involution (as the functions of I_1, I_2, I_3). They determine the two-dimensional tori. It is easier to see this in the action-angle variables 'compatible with the structure of trajectories' (CST).

We shall call the action-angle variables CST variables if the Hamiltonian is a function of a one-action variable only:

$$H = H(I_1).$$

Thus the equations of motion in CST variables take the form

$$\dot{I}_1 = 0, \qquad \dot{w}_1 = dH/dI_1,$$

$$\dot{I}_2 = 0, \qquad \dot{w}_2 = 0.$$

The variable w_1 describes a linear motion along a closed curve and w_2 is an infinitely-many-valued integral of motion. As the third single-valued integral of motion one can take $I_3 = \sin w_2$.

It is easy to show that CST variables exist for every degenerate Hamiltonian system with two degrees of freedom. Namely, let $H = H(I_1/l_1 + I_2/l_2)$ with l_1, l_2 mutually prime; then one can always choose integers k_1, k_2 satisfying $l_2k_1 - l_1k_2 = 1$ and define new action variables as

$$J_1 = l_2I_1 + l_1I_2, \qquad J_2 = k_2I_1 + k_1I_2. \tag{*}$$

They are CST variables since $H = H[(1/l_1l_2)J_1]$.

The freedom of choice of CST variables for a fixed Hamiltonian is described by canonical transformations of the form

$$\begin{aligned} J_1 &= f_1(I_1), & u_1 &= (1/f_1')w_1 + g_1(I_1, I_2, w_2), \\ J_2 &= f_2(I_1, I_2, I_3), & u_2 &= g_2(I_1, I_2, w_2). \end{aligned} \tag{2}$$

A one-to-one description of the phase space requires

$$\begin{aligned} &f_1' = \pm 1, \\ &g_1(I_1, I_2, w_2 + 2\pi) - g_1(I_1, I_2, w_2) = 2n\pi, \\ &g_2(I_1, I_2, w_2 + 2\pi) - g_2(I_1, I_2, w_2) = \pm 2\pi, \end{aligned} \tag{**}$$

and from the canonical Poisson-bracket relations it follows that

$$\begin{aligned} &\frac{\partial f_2}{\partial w_2}\frac{\partial g_1}{\partial I_2} - \frac{\partial f_2}{\partial I_2}\frac{\partial g_1}{\partial w_2} = \frac{1}{f_1'}\frac{\partial f_2}{\partial I_1}, \\ &\frac{\partial f_2}{\partial w_2}\frac{\partial g_2}{\partial I_2} - \frac{\partial f_2}{\partial I_2}\frac{\partial g_2}{\partial w_2} = -1, \\ &\frac{\partial g_1}{\partial I_2}\frac{\partial g_2}{\partial w_2} - \frac{\partial g_1}{\partial w_2}\frac{\partial g_2}{\partial I_2} = \frac{1}{f_1'}\frac{\partial g_2}{\partial I_1}. \end{aligned} \tag{**}$$

Thus, the canonical transformations (2) complemented by the transformations on fixed tori (1) represent the whole freedom of choice of the action-angle variables for a degenerate Hamiltonian system with two degrees of freedom. Moreover, as in CST variables the Hamiltonian has the form $H = f(\pm I_1)$ and transformations to all other action variables are of the form $I_1 = l_2J_1 + l_1J_2$, where l_1, l_2 are mutually prime, we arrive at the following conclusion. The dependence of the Hamiltonian on the action-angle variables is of the form $H = f[\pm(l_2J_1 + l_1J_2)]$ where l_1, l_2 are arbitrary mutually prime integers and f is a fixed function of one variable.

The formulae for the canonical transformations (2) as well as the above conclusion can be simply generalised for completely degenerate Hamiltonian systems with more than two degrees of freedom. It is sufficient to introduce CST variables through the application of the transformation (*) to various pairs of action variables in the way illustrated by the following example.

Let us take the Hamiltonian $H = H(l_1I_1 + l_2I_2 + l_3I_3)$ for the system with three degrees of freedom. New action variables can be introduced by the formulae

$$J_1 = l_1'I_1 + l_2'I_2, \qquad J_2 = k_1I_1 + k_2I_2, \qquad J_3 = I_3,$$

where $l'_1k_2 - l'_2k_1 = 1$, l'_1, l'_2 are mutually prime and $l_k = ll'_k$. We define again

$$J'_1 = lJ_1 + l_3J_3 = l_1I_1 + l_2I_2 + l_3I_3,$$
$$J'_2 = J_2 = k_1I_1 + k_2I_2,$$
$$J'_3 = k_3J_3 + kJ_1 = kl'_1I_1 + kl'_2I_2 + k_3I_3 \qquad \text{where } lk_3 - l_3k = 1.$$

The Hamiltonian in the variables J'_k is of the form $H = H(J'_1)$, and therefore J'_k are CST variables.

In the next two examples we show the simplest transformations satisfying (**) and discuss their relationship with the separability of the Hamiltonian in various coordinate systems.

Example 1. For the simplest interesting functions $f_1 = I_1, f_2 = I_3 = \sin w_2$, the equations (**) have the following solutions satisfying the conditions of periodicity:

$$g_1 = nw_2, \qquad g_2 = \pm w_2 + (nI_1 - I_2)/\cos w_2.$$

Thus, the new action-angle variables are of the form

$$J_1 = I_1, \qquad u_1 = w_1 + nw_2,$$
$$J_2 = \sin w_2, \qquad u_2 = \pm w_2 + (nI_1 - I_2)/\cos w_2.$$

Example 2. The separation of the Hamiltonian of the harmonic oscillator in the cartesian coordinates (x, y) gives the following action-angle variables:

$$I_x = \frac{1}{2\omega}p_x^2 + \frac{\omega}{2}x^2, \qquad w_x = \sin^{-1}\left(\frac{\omega}{2I_x}\right)^{1/2} x,$$
$$I_y = \frac{1}{2\omega}p_y^2 + \frac{\omega}{2}y^2, \qquad w_y = \sin^{-1}\left(\frac{\omega}{2I_y}\right)^{1/2} y,$$

where $H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\omega^2(x^2 + y^2)$. Therefore the CST variables have the form

$$I_1 = I_x + I_y, \qquad w_1 = w_y, \qquad I_2 = I_x, \qquad w_2 = w_x - w_y.$$

By separation of the Hamiltonian in the new, rotated coordinates

$$u = x \sin\theta + y \cos\theta, \qquad v = -x\cos\theta + y \sin\theta,$$

we obtain CST variables connected with I_1, I_2 by the transformation

$$\begin{aligned} I'_1 &= f'_1(I_1) = I_1, \\ I'_2 &= f'_2(I_1, I_2) = I_1\cos^2\theta - I_2\cos 2\theta + (I_1I_2 - I_2^2)^{1/2}(\cos w_2)\sin 2\theta. \end{aligned} \tag{2'}$$

Likewise, the separation in polar coordinates gives CST variables connected with I_1, I_2 by the transformation

$$I''_1 = f''_1(I_1) = I_1, \qquad I''_2 = f''_2(I_1, I_2) = \tfrac{1}{2}I_1 - (I_1I_2 - I_2^2)^{1/2}\sin w_2. \tag{2''}$$

This means that for the functions f_1, f_2 defined by the transformations (2′), (2″) there exist solutions of equations (**) satisfying the necessary conditions of periodicity. This fact will be useful in the next section.

5. Relationship of the action-angle variables to the generators of the symmetry group

In this section we shall use the SU(2) algebra of integral-generators of the symmetry group to obtain the action variables.

Duimio and Pauri (1967) have presented a general technique for the construction of the SU(n) algebra of the integrals of motion from the action-angle variables. It is applicable for every completely degenerate bounded Hamiltonian system with n degrees of freedom.

In the case $n=2$, if the Hamiltonian is of the form $H=H(I_1/l_1+I_2/l_2)$ where $l_1I_1>0$, $l_2I_2>0$, the most general realisation of the generators of the symmetry group SU(2) (up to an arbitrary single-valued canonical transformation) uniquely determining each trajectory is defined by the global and univalent integrals

$$A_1=\left(\frac{I_1}{l_1}\frac{I_2}{l_2}\right)^{1/2}\sin(l_1w_1-l_2w_2),\qquad A_2=\left(\frac{I_1}{l_1}\frac{I_2}{l_2}\right)^{1/2}\cos(l_1w_1-l_2w_2),$$
$$A_3=\tfrac{1}{2}(I_2/l_2-I_1/l_1).\tag{3}$$

The integrals A_i satisfy the commutation relations $\{A_i,A_j\}=\varepsilon_{ijk}A_k$ of the SU(2) algebra. The Casimir operator is $K=A_1^2+A_2^2+A_3^2=\frac{1}{4}(I_1/l_1+I_2/l_2)^2$. In view of the generality of the Duimio and Pauri (1967) construction (cf Onofri and Pauri 1969), each set of the generators of SU(2) can be determined by formulae (3) from some action-angle variables. Thus by inverting formulae (3) one can recover the action variables:

$$I_1/l_1=(K)^{1/2}-A_3,\qquad I_2/l_2=(K)^{1/2}+A_3.$$

The cyclic form of the commutation relations $\{A_i,A_j\}=\varepsilon_{ijk}A_k$ suggests that the different action variables can be defined by

$$J_1/l_1=(K)^{1/2}-A_k,\qquad J_2/l_2=(K)^{1/2}+A_k,\qquad k=1,2.$$

To prove that J_i are indeed the 'actions' (i.e. can complement angles) we shall use CST variables. In agreement with the previous section, every Hamiltonian can be described in certain action-angle variables as $H=H(I_1+I_2)$.

Let us introduce CST variables by

$$J_1=I_1+I_2,\qquad u_1=w_2,\qquad J_2=I_1,\qquad u_2=w_1-w_2;$$

then the generators of SU(2) have the form

$$A_1=(J_1J_2-J_2^2)^{1/2}\sin u_2,\qquad A_2(J_1J_2-J_2^2)^{1/2}\cos u_2,\qquad A_3=\tfrac{1}{2}J_1-J_2,$$

and inverse relations yield

$$J_1=2(K)^{1/2},\qquad J_2=(K)^{1/2}-A_3.$$

According to our guess, we define new CST variables as

$$J_1'=2(K)^{1/2}=J_1,\qquad J_2'=(K)^{1/2}-A_1=\tfrac{1}{2}J_1-(J_1J_2-J_2^2)^{1/2}\sin u_2.$$

The results of example 2 (2′) ensure that the above formulae really determine the transformation to CST variables. This is also true for the transformation defined by

$$J_1''=2(K)^{1/2}=J_1,\qquad J_2''=(K)^{1/2}-A_2=\tfrac{1}{2}J_1-(J_1J_2-J_2^2)^{1/2}\cos u_2,$$

as the mutual interchange of $\sin u_2$ and $\cos u_2$ does not change the periodicity of solutions of the equations (**).

The above considerations present the method of obtaining the action-angle variables. Namely, one can find the generators of the SU(2) symmetry group and obtain the action variables by the formulae

$$I_1/l_1 = (K)^{1/2} - A_k, \qquad I_2/l_2 = (K)^{1/2} + A_k, \qquad k = 1, 2, 3.$$

It is necessary to choose the integers l_1, l_2 properly in order to ensure periodicity of the corresponding angle variables. This procedure allows us to connect with every SU(2) algebra of integrals three sets of action-angle variables. Thus, in order to construct new action variables, one can start from the algebra of integrals A_i and look for new sets of integrals given by functions $f_k(A_i)$. The commutation relations $\{f_k, f_l\} = \varepsilon_{kls} f_s$ provide us with the following system of differential equations for the generators:

$$\sum_{ijk} \left(\frac{\partial f_r}{\partial A_i} \frac{\partial f_s}{\partial A_j} - \frac{\partial f_r}{\partial A_j} \frac{\partial f_s}{\partial A_i} \right) A_k = f_t, \tag{4}$$

where $i, j, k, r, s, t = 1, 2, 3$.

The cyclic form of this equation ensures that the general solution is given by the functions f_i of the first order in A_j. Therefore the Casimir operator $f_1^2 + f_2^2 + f_3^2 = g(K)$, given by some function of the old one, has to satisfy $f_1^2 + f_2^2 + f_3^2 = cK$, where c is some integer constant. We do not know a general solution for the system of equations (4). As a simple solution (preserving the Casimir operator of the algebra) we shall consider the rotations by an angle θ around the A_1 axis in the space of integrals. They lead to the following new generators:

$$B_1 = A_1, \qquad B_2 = A_2 \cos\theta + A_3 \sin\theta, \qquad B_3 = -A_2 \sin\theta + A_3 \cos\theta.$$

Identical generators can be obtained from formulae (3) applied to action-angle variables of the harmonic oscillator corresponding to separation coordinates rotated by an angle $\frac{1}{2}\theta$. It is straightforward to generalise this rotation by allowing angle θ to be dependent on the generators A_i. Take

$$B_1 = A_1, \qquad B_2 = A_2 \cos f(A_1, A_2, A_3) + A_3 \sin f(A_1, A_2, A_3),$$
$$B_3 = -A_2 \sin f(A_1, A_2, A_3) + A_3 \cos f(A_1, A_2, A_3);$$

then by virtue of (4), $f(A_1, A_2, A_3) = \hat{f}(A_1, A_2^2 + A_3^2)$.

However, there are solutions of equations (4) which do not preserve the Casimir operator. Let us take two algebras of integrals for the harmonic oscillator obtained by the separation in cartesian coordinates (x, y) and coordinates defined by $u = x - y$, $v = x + y$. They are connected by the transformation

$$B_1 = -\tfrac{1}{2} A_1, \qquad B_2 = \frac{A_2 A_3}{(A_2^2 + A_3^2)^{1/2}}, \qquad B_3 = \frac{A_2^2 - A_3^2}{(A_2^2 + A_3^2)^{1/2}},$$

which does not preserve the Casimir operator: $B_1^2 + B_2^2 + B_3^2 = (A_1^2 + A_2^2 + A_3^2)/4$.

It is simple to generalise the above considerations for completely degenerate Hamiltonian systems with more than two degrees of freedom. Formulae for the generators of the SU(n) symmetry group given by Duimio and Pauri (1967), i.e.

$$N_{ij}=2\left(\frac{I_i}{l_i}\frac{I_j}{l_j}\right)^{1/2}\cos(l_iw_i-l_jw_j),\qquad N_{ii}=2\left(\frac{I_i}{l_i}-\frac{1}{n}\sum_k\frac{I_k}{l_k}\right),$$

$$M_{ij}=2\left(\frac{I_i}{l_i}\frac{I_j}{l_j}\right)^{1/2}\sin(l_iw_i-l_jw_j),$$

allow us to enlarge the prescription for obtaining 'actions'. It is necessary to solve the system of algebraical equations

$$\frac{I_iI_j}{l_il_j}=(M_{ij}^2+N_{ij}^2)/4\qquad\text{(for } n\geqslant 3).$$

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